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على وحدات الضرب المتدرجة S

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On graded S −multiplication modules

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الملخص:

في هذا البحث، نقدم مفهوم وحدات الضرب المتدرجة S والتي تعد تعميمًا لوحدات الضرب المتدرجة. تم إثبات العديد من النتائج المتعلقة بوحدات الضرب المتدرجة S . ندرس الوحدات الفرعية الأولية المتدرجة S في وحدات الضرب المتدرجة S. أيضًا، نقوم بتعميم نظرية التجنب الأولي لوحدات الضرب إلى وحدات الضرب المتدرجة S. نحن نصف وحدات الضرب المتدرجة من حيث وحدات الضرب المتدرجة S.

Abstract:

In this paper, we introduce the concept of graded $S-$multiplication modules which are a generalization of graded multiplication modules. Several results concerning graded $S-$multiplication modules are proved. We study graded $S-$prime submodules in graded $S-$multiplication modules. Also, we generalize prime avoidance lemma for multiplication modules to graded $S-$multiplication modules. We characterize graded multiplication modules in terms of graded $S-$multiplication modules.

Keywords: Graded $S-$multiplication module; graded multiplication module; graded prime module; graded torsion module.

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1 Introduction

Graded ring theory plays an important role in commutative algebra. Graded rings contribute to the development of algebra applications in algebraic geometry, functional analysis and theoretic physics. Recently, extensive researches have been done on rings with group-graded structure, see for example [1, 2, 3, 10, 11, 14]. The notion of graded multiplication modules was studied by many authors, see for example [4, 6, 9, 17]. The motivation of writing this paper is two folded:

(i) To extend the concept of graded multiplication modules to the concept of graded $S-$multiplication modules.

(ii) To determine when a graded module is graded $S-$multiplication modules. The remains of this paper is organized as follows:

Section 2 concerns some basic definitions and results in the sequel of this paper. In section 3, the main results concerning graded $S-$multiplication modules will be given. Section 4 concerns the conclusion.

2 Preliminary Notes

In this section we state some basic concepts and results related to graded ring theory. We hope that this will improve the readability and understanding of this paper.

Definition 2.1 *[12] Let G be a group with identity e and R a commutative ring with identity 1_R. Then, R is said to be a $G-$graded ring if there exist additive subgroups R_g of R indexed by elements $g \in G$ such that $R = \bigoplus_{g \in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. If $R_g R_h = R_{gh}$, the ring is called strongly graded ring.*

Consider $supp(R) = \{g \in G: R_g \neq 0\}$. An element x of R has a unique decomposition as $x = \sum_{g \in G} x_g$ for all $g \in G$. Also, we write $h(R) = \bigcup_{g \in G} R_g$. Moreover, R_e is a subring of R and $1_R \in R_e$. If an element of R belongs to $h(R)$, then it is called homogeneous and any element $x_g \in R_g$

is said to have degree g.

Definition 2.2 *[12] Let* $R = \bigoplus_{g \in G} R_g$ *be a* G *−graded ring. An ideal* I *of* R *is said to be a graded ideal of* R *if* $I = \bigoplus_{g \in G} (I \cap R_g)$.

Clearly, $\bigoplus_{g \in G} (I \cap R_g) \subseteq I$ and hence I is a graded ideal of R if $I \subseteq \bigoplus_{g \in G} (I \cap R_g)$. Moreover R/I becomes a G −graded ring with g −component $(R/I)_g = (R_g + I)/I$ for $g \in G$.

Definition 2.3 *[12] Let* R *be a* G *−graded ring and* M *an* R *−module. We say that* M *is a graded* R *−module if there exists a family of subgroups* $\{M_g\}_{g \in G}$ *of* M *such that* $M = \bigoplus_{g \in G} M_g$ *(as abelian groups) and* $R_g M_h \subseteq M_{gh}$ *for all* $g, h \in G$. *If* $R_g M_h = M_{gh}$, *the* R *−mould* M *is called strongly graded* R *−module.*

Consider $supp(M) = \{g \in G : M_g \neq 0\}$. Here $R_g M_h$ denotes the additive subgroups of M consisting of all finite sums of elements $r_g s_h$ with $r_g \in R_g$ and $s_h \in M_h$. Also, we write $h(M) = \bigcup_{g \in G} M_g$. If an element of M belongs to $h(M)$, then it is called homogeneous and any element $x_g \in M_g$ is said to have degree g. It is clear that M_g is an R_e −submodule of M for all $g \in G$.

Definition 2.4 *[12] Let* $M = \bigoplus_{g \in G} M_g$ *be a* G *−graded* R *−module and* N *a submodule of* M. *Then,* N *is said to be a graded submodule of* M *if* $N = \bigoplus_{g \in G} N_g$ *where* $N_g = N \cap M_g$ *for* $g \in G$. *In this case,* N_g *is called the* g *−component of* N *for* $g \in G$. *Moreover,* M/N *becomes a* G *−graded module with* g *−component* $(M/N)_g = (M_g + N)/N$ *for* $g \in G$.

Definition 2.5 *[15] Let* I *be a graded ideal of a* G *−graded ring* R. *Then,* I *is said to be a graded prime ideal if* $I \neq R$; *and whenever* $ab \in I$, *we have* $a \in I$ *or* $b \in I$, *where* $a, b \in h(R)$.

Definition 2.6 *[5] Let* R *be a* G *−graded ring and* M *a graded* R *−module. A graded submodule* N *of* M *is said to be a graded prime submodule of* M *if* $N \neq M$; *and whenever* $r \in h(R)$ *and* $m \in h(M)$ *with* $rm \in N$, *then either* $m \in N$ *or* $r \in (N :_R M)$.

Definition 2.7 *[12] Let* R *be a* G *−graded ring. A nonzero graded* R *−module* M *is said to be a graded prime module if* $Ann_R(M) = Ann_R(N)$ *for every nonzero graded submodule* N *of* M.

Definition 2.8 *[12] Let* R *be a* G *−graded ring. A nonempty* $S \subseteq h(R)$ *is said to be a multiplicatively closed subset of* R *if* (i) $0 \notin S$, (ii) $1 \in S$, (iii) $ab \in S$ *for all* $a, b \in S$.

Definition 2.9 *[16] let* R *be a* G *−graded ring,* $S \subseteq h(R)$ *a multiplicatively closed subset of* R *and* M *be a graded* R *−module. A graded submodule* N *of* M *with* $(N :_R M) \cap S = \phi$ *is said to be a graded* S *−prime submodule of* M *if there exists a fixed* $x \in S$ *such that whenever* $rm \in N$ *for some* $r \in h(R)$ *and* $m \in h(M)$, *then either* $xr \in (N :_R M)$ *or* $xm \in N$. *In particular, a graded ideal* P *of* R *is said to be a graded* S *−prime if* P *is a graded* S *−prime submodule of* M.

Definition 2.10 *[12] Let* R *be a* G *−graded ring. A graded* R *−module* M *is said to be graded finitely generated if* $M = R_{m_1} + R_{m_2} + \ldots + R_{m_n}$ *for some* $m_1, m_2, \ldots, m_n \in h(M)$. M *is called a graded cyclic if it can be generated by a single element i.e., there exists* $x \in h(M)$ *such that* $M = Rx$.

Definition 2.11 *[12] Let* R *be a*

G −graded ring and $M, \overline{M}$ graded R −modules. Then, an R −homomorphism $f: M \longrightarrow \overline{M}$ is said to be a graded R −homomorphism if for all $m, n \in M$;

(i) $f(m+n) = f(m) + f(n)$;

(ii) $f(rm) = rf(m)$ for any $r \in R$ and $m \in M$;

(iii) $f(M_g) \subseteq \overline{M_g}$ for all $g \in G$.

Remark 2.12 *[13] Let R be a G −graded ring, M a graded R −module, P a graded ideal of R and N a graded submodule of M. Then,*

(i) $Ann_R(M) = (0:_R M) = \{r \in R: rM = 0\}$ is a graded ideal of R.

(ii) $(0:_M P) = \{m \in M: mP = 0\}$ is a graded submodule of M.

(iii) $Ann_R(N) = (0:_R N) = \{r \in R: rN = 0\}$ is a graded ideal of R.

Definition 2.13 *[1] Let R be a G −graded ring. A graded R −module M is said to be a graded multiplication module if for every graded submodule N of M, there exists a graded ideal I of R such that $N = IM$.*

3 Results and Discussion

Our starting point is the following definition.

Definition 3.1 *Let R be a G −graded ring, M a graded R −module and $S \subseteq h(R)$ a multiplicatively closed subset of R. M is is said to be a graded S −multiplication module if for each graded submodule N of M, there exist $x \in S$ and a graded ideal P of R such that $xN \subseteq PM \subseteq N$.*

We define the graded ring R to be a graded S −multiplication ring if it is a graded S −multiplication module over itself.

Remark 3.2

(i) One can easily see that a graded R −module M is a graded S −multiplication module if and only if for each graded submodule N of M, there exists $x \in S$ such that $xN \subseteq (N:_R M)M \subseteq N$.

(ii) If $S \subseteq h(R)$ is a multiplicatively closed subset of R and M is a graded R −module with $Ann_R(M) \cap S \neq \phi$, then M is a graded S −multiplication module.

(iii) Every graded multiplication module is also graded S −multiplication module.

The converse is true only, when $S \subseteq U(R)$, where $U(R)$ denotes the set of all units in R.

Now, we give an example of a graded S −multiplication module which is not graded multiplication module.

Example 3.3 *Consider $R = \mathbb{Z}$, $G = \mathbb{Z}_2$, $R_0 = \mathbb{Z}$ and $R_1 = \{0\}$. Then, R is a G −graded ring. Let M be the graded $\mathbb{Z}$ − module, $M = \{\alpha: \alpha = \frac{m}{p^n} + \mathbb{Z}, m \in h(\mathbb{Z}), n \geq 0\}$, where p is a prime number. Now, $Ann_{\mathbb{Z}}(M) = 0$. Thus, M is a graded faithful $\mathbb{Z}$ −module. Now, let $S = \{p^t: t \in \mathbb{N} \cup \{0\}\} \subseteq h(\mathbb{Z})$ is a multiplicatively closed set. One can see that all graded proper submodules of M are of the form $T_{n_0} = \{\alpha: \alpha = \frac{m}{p^{n_0}} + \mathbb{Z}, m \in h(\mathbb{Z})\}$ for some $n_0 \in \mathbb{N} \cup \{0\}$. Assume that $n_0 \geq 1$. Then, $(T_{n_0}:_{\mathbb{Z}} M) = 0$ and thus $T_{n_0} \neq (T_{n_0}:_{\mathbb{Z}} M)M = 0_M$. Thus, M is not graded multiplication module. Now, take $x = p^{n_0} \in S$. Then, one can easily see that $xT_{n_0} = 0 \subseteq (T_{n_0}:_{\mathbb{Z}} M)M \subseteq T_{n_0}$. Thus, the graded $\mathbb{Z}$ −module M is a graded faithful S −multiplication module.*

We need the following Lemma.

Lemma 3.4 *[7] Let R be a G −graded ring, M be a graded R −module. Then, the following assertions hold.*

(i) If I and J are graded ideals of R, then $I + J$ and $I \cap J$ are graded ideals of R.

(ii) If N is a graded submodule of M, $r \in h(R)$, $x \in h(M)$ and I is a graded ideal of R, then Rx, IN and rN are graded submodules of M.

(iii) If N and K are graded submodules of M, then $N + K$ and $N \cap K$ are also graded submodules of M and $(N :_R M) = \{r \in R : rM \subseteq N\}$ is a graded ideal of R.

Now, we recall the following definition.

Definition 3.5 *[10] Let R be a G − graded ring, $S \subseteq h(R)$ a multiplicatively closed subset of R and M a graded R −module. Then, $S^{-1}M$ is a graded $S^{-1}R$ − module where,*

The ring of fraction is defined by:

$$(S^{-1}R)_g = \{\frac{r}{x} : r \in h(R), x \in S \; and \; g = deg\, r - deg\, x \; for \; all \; g \in G\}$$

The quotient module M is thus defined by:

$$(S^{-1}M)_g = \{\frac{m}{x} : m \in h(M), x \in S \; and \; g = deg\, m - deg\, x \; for \; all \; g \in G\}$$

The saturation $S^\star$ of S is defined by:

$$S^\star = \{x \in h(R) : x \; divides \; s \; for \; some \; s \in S\} \subseteq h(R)$$

is a multiplicatively closed subset of R containing S.

Also, S is called a saturated multiplicatively closed set if $S = S^\star$. Note that $S^\star$ is always a saturated multiplicatively closed set.

Theorem 3.6 *Let R be a G −graded ring, $S \subseteq h(R)$ a multiplicatively closed subset of R and M a graded R −module. Then, the following assertions hold.*

(i) Let $S_1 \subseteq h(R)$ and $S_2 \subseteq h(R)$ be two multiplicatively closed subsets of R with $S_1 \subseteq S_2$. If M is a graded S_1 −multiplication module, then M is also a graded S_2 −multiplication module.

(ii) M is a graded S −multiplication module if and only if M is a graded $S^\star$ −multiplication module.

Proof. (i) It is clear.

(ii) Assume that M is a graded S −multiplication module. Since $S \subseteq S^\star$, then the result follows from part (i). Conversely; assume that M is a graded $S^\star$ −multiplication module and let N be a graded submodule of M. Then, there exists $x \in S^\star$ such that $xN \subseteq (N :_R M)M$. Choose $r \in h(R)$ with $xr \in S$. Thus, $(xr)N \subseteq (N :_R M)M \subseteq N$. Therefore, M is a graded S −multiplication module.

Now, we have the following notation.

Notation:

(i) Let R be a G −graded ring and P a graded prime ideal of R. Then, $S = h(R) - P$ is a multiplicatively closed subset and we denote the the graded ring of fraction $S^{-1}R$ of R by R_P. If R_P is graded local ring with unique graded maximal ideal $S^{-1}P$, then we denote it by PR_P.

(ii) The set of all graded prime ideals of R is dented by $GSpec(R)$ and the set of all graded maximal ideals of R is denoted by $GMax(R)$.

Theorem 3.7 *Let R be a G −graded ring and M a graded R −module. Then, the following assertions are equivalent.*

(i) M is a graded multiplication module.

(ii) M is a graded $(h(R) - I)$ −multiplication module for each $I \in GSpec(R)$.

(iii) M is a graded $(h(R) - L)$ −multiplication module for each $L \in GMax(R)$.

(iv) M is a graded $(h(R) - L)$ −multiplication module for each $L \in GMax(R)$ with $M_L \neq 0_L$.

Proof. $(i) \Longrightarrow (ii)$ Assume that M is a graded multiplication module and let $I \in GSpec(R)$. Then, $(h(R) - I) \subseteq h(R)$ is a multiplicatively closed subset of R and thus M is a graded $(h(R) - I)$ −multiplication module.

$(ii) \Longrightarrow (iii)$ Since every graded maximal ideal is graded prime, then the result follows from part (ii).

$(iii) \Longrightarrow (iv)$ It is clear.

$(iv) \Longrightarrow (i)$ Let N be a graded submodule of M. Now, let L be a graded maximal ideal of R with $M_L \neq 0_L$. Since M is a graded $(h(R) - L)$ −multiplication module, then there exists $a \in h(R) - L$ such that $aN \subseteq (N:_R M)M$. Thus, $N_L = (aN)_L \subseteq ((N:_R M)M)_L \subseteq N_L$. If $M_L = 0_L$, then $N_L = ((N:_R M)M)_L$. Thus, we have $N_L = ((N:_R M)M)_L$ for each graded maximal ideal L of R and thus $N = (N:_R M)M$. Therefore, M is a graded multiplication module.

Theorem 3.8 *Let R be a G −graded ring, $S \subseteq h(R)$ a multiplicatively closed subset of R, M_1 and M_2 graded R −modules, and $\psi: M_1 \longrightarrow M_2$ a graded R −homomorphisim with $rKer(\psi) = 0$ for some $r \in S$. Then, the following assertions hold.*

(i) If ψ is a graded R −epimorphism and M_1 is a graded S −multiplication module, then M_2 is a graded S −multiplication module.

(ii) If M_2 is a graded S −multiplication module, then M_1 is a graded S −multiplication module.

Proof. (i) Assume that M_1 is a graded S −multiplication module and let N_2 be a graded submodule of M_2. Then $N_1 = \psi^{-1}(N_2)$ is a graded submodule of M_1. Now, M_1 is a graded S −multiplication module, implies that there exists $x \in S$ such that $xN_1 \subseteq (N_1:_R M_1)M_1 \subseteq N_1$. Thus, $\psi(xN_1) \subseteq \psi((N_1:_R M_1)M_1) \subseteq \psi(N_1)$. Thus, $xN_2 = x\psi(N_1) \subseteq (N_1:_R M_1)\psi(M_1) = (N_1:_R M_1)M_2 \subseteq N_2$. Thus, $xN_2 \subseteq (N_1:_R M_1)M_2 \subseteq N_2$. Therefore, M_2 is a graded S −multiplication module.

(ii) Let N_1 be a graded submodule of M_1. As $\psi(M_1)$ is a graded S −multiplication module, then there exist $x \in S$ and a graded ideal P of R with $x\psi(N_1) \subseteq P\psi(M_1) \subseteq \psi(N_1)$. Thus, $xN_1 + Ker(\psi) \subseteq PM_1 + Ker(\psi) \subseteq N_1 + Ker(\psi)$. Multiply by r and noting that $rKer(\psi) = 0$, we have $(xr)N_1 \subseteq (rP)M_1 \subseteq rN_1 \subseteq N_1$. Therefore, M_1 is a graded S −multiplication module.

Corollary 3.9 *Let R be a $G -$ graded ring, $S \subseteq h(R)$ a multiplicatively closed subset of R, M a graded R −module and N a graded submodule of M. Then, the following assertions hold.*

(i) If M is a graded S −multiplication module, then M/N is a graded S −multiplication module.

(ii) If M/N is a graded S −multiplication module and $rN = 0$ for some $r \in S$, then M is a graded S −multiplication module.

Proof. Follows directly from Theorem 3.8.

Theorem 3.10 *Let R be a $G -$ graded ring, $S \subseteq h(R)$ and $T \subseteq h(R)$ multiplicatively closed subsets of R. Let $\overline{S} = \{\frac{x}{1} \in T^{-1}R : x \in S\} \subseteq h(T^{-1}R)$ be a multiplicatively closed subsets of $T^{-1}R$. If M is a graded S −multiplication R −module, then $T^{-1}M$ is a graded $\overline{S}$ −multiplicaton $T^{-1}R$ −module. Thus, if $S \subseteq T^{*}$, then $T^{-1}M$ is a graded multiplication $T^{-1}R$ −module. Hence, $S^{-1}M$ is a graded multiplication $S^{-1}R$ −module.*

Proof. Assume that M is a graded S −multiplication R −module and let N be a graded $T^{-1}R$ −submodule of $T^{-1}M$. Then, $N = T^{-1}N_1$ for some graded submodule N_1 of M. As M is a graded S −multiplication R −module, then there exist $x \in S$ and a graded ideal P of R with $xN_1 \subseteq PM \subseteq N_1$. Then, $\frac{x}{1}N = T^{-1}(xN_1) \subseteq (T^{-1}P)(T^{-1}M) \subseteq T^{-1}N_1 = N$. Thus, $T^{-1}M$ is a graded $\overline{S}$ −multiplication $T^{-1}R$ −module. If $S \subseteq T^{\star}$, then $\overline{S} \subseteq U(T^{-1}R)$. Therefore, $T^{-1}M$ is a graded multiplication $T^{-1}R$ −module.

Recall that from [10] if R be a G −graded ring and $S \subseteq h(R)$ be a multiplicatively closed subset of R, then S is said to satisfy the maximal multiple condition if there exists $x \in S$ such that t divides x for each $t \in S$.

Corollary 3.11 *Let R be a G −graded ring, $S \subseteq h(R)$ a multiplicatively closed subset of R satisfying the maximal multiple condition and M a graded R −module. Then M is a graded S −multiplication module if and only if $S^{-1}M$ is a graded multiplication $S^{-1}R$ −module.*

Proof. The first implication follows directly from Theorem 3.10. For the converse, assume that $S^{-1}M$ is a graded multiplication $S^{-1}R$ −module and let N be a graded submodule of M. Now $S^{-1}M$ is a graded multiplication $S^{-1}R-$ module implies that $S^{-1}N = (S^{-1}P)(S^{-1}M) = S^{-1}(PM)$ for some a graded ideal P of R. Choose $x \in S$ such that t divides x for each $t \in S$. Now, for each $m \in N \cap h(M)$, we get $\frac{m}{1} \in S^{-1}N = S^{-1}(PM)$. Thus, there exists $t \in S$ such that $tm \in PM$ and thus, $xm \in PM$. Thus, $xN \subseteq PM$. On the other hand, a similar argument shows that $xPM \subseteq N$. Therefore, $x^2N \subseteq (xP)M \subseteq N$. Hence, M is a graded S −multiplication module.

Theorem 3.12 *Let R be a $G-$ graded ring, $S \subseteq h(R)$ a multiplicatively closed subset of R, M a graded S −multiplication R −module and N a graded submodule of M. Then, N is a graded S −prime submodule of M if and only if $(N:_R M)$ is a graded S −prime ideal of R.*

Proof. The first implication follows directly from ([16], Proposition 2.6). For the converse, assume that $(N:_R M)$ is a graded S −prime ideal of R. Thus, there exists $x \in S$ such that whenever $rm \in (N:_R M)$ for some $r \in h(R)$ and $m \in h(M)$, then either $xr \in (N:_R M)$ or $xm \in (N:_R M)$. Moreover by ([16], Lemma 4.2), we have $(N:_R \overline{x}) \subseteq (N:_R x)$ for all $\overline{x} \in S$. Suppose that $y \in h(R)$ and $t \in h(M)$ with $yt \in N$ and choose $a \in S$ with $aRt \subseteq (Rt:_R M)M$. For $r \in (Rt:_R M) \cap h(R)$, we have $yr \in (N:_R M)$; so $xy \in (N:_R M)$ or $xr \in (N:_R M)$. If $xy \in (N:_R M)$, we are done. So, assume that $xr \in (N:_R M)$ for all $r \in (Rt:_R M) \cap h(R)$. Then, $x(Rt:_R M) \subseteq (N:_R M)$. Thus, $x(Rt:_R M)M \subseteq (N:_R M)M \subseteq N$. Thus, $xaRt \subseteq N$. Thus, $y \in (N:_R xa) \subseteq (N:_R x)$. Hence, $xt \in N$.

Now, we introduce the following definition.

Definition 3.13 *Let R be a G −graded ring, $S \subseteq h(R)$ a multiplicatively closed subset of R and M a graded R −module. Then, M is called a graded S −cyclic module if there exist $x \in S$ and $m \in h(M)$ with $xM \subseteq Rm \subseteq M$. For a graded prime ideal Q of R, M is called graded Q −cyclic if M is graded $(h(R) - Q)$ −cyclic.*

Theorem 3.14 *Let R be a $G-$ graded ring, $S \subseteq h(R)$ a multiplicatively closed subset of R and M a graded R −module. If M is a graded S −cyclic module, then M is a graded S −multiplication module.*

Proof. Assume that M is a graded S −cyclic R −module. Then, there exist $x \in S$ and $m \in h(M)$ with $xM \subseteq Rm \subseteq M$. Let N be a graded submodule of M. Then, $xN \subseteq xM \subseteq Rm$ and thus, $xN = PRm$ for some ideal P of R. Thus, $x^2N = xPRm \subseteq xPM \subseteq PRm = xN \subseteq N$. Now, $x^2 \in S$ and xP is a graded ideal of R. Thus, M is a graded S −multiplication module.

Theorem 3.15 *Let R be a G − graded ring, Q a graded prime ideal of R and M a graded R −module with $M_Q \neq 0_Q$. If M is a graded $(Q \cap h(R))$ −multiplicaton module, then M is a graded $(Q \cap h(R))$ −cyclic.*

Proof. Assume that M is a graded $(Q \cap h(R))$ −multiplication module with $M_Q \neq 0_Q$. Then, by Theorem 3.10, M_Q is a graded R_Q −multiplication module. Thus, M_Q is a graded cyclic module. Hence, $Q_QM_Q \neq M_Q$. Choose $m_0 \in h(M)$ with $R_Qm_0 \nsubseteq 0_QM_Q$. Now, M is graded $(Q \cap h(R))$ −multiplication module implies that there exist $x \in h(R) - Q$ and a graded ideal P of R with $xRm_0 \subseteq PM \subseteq Rm_0$. Also, we have $P \nsubseteq Q$. For if $P \subseteq Q$, then $(Rm_0)_Q = (xRm_0)_Q \subseteq P_QM_Q \subseteq Q_QM_Q$, a contradiction. Choose $y \in P \cap h(R) - Q$. Then, $yM \subseteq PM \subseteq Rm_0$. Therefore, M is graded $(Q \cap h(R))$ −cyclic.

Corollary 3.16 *Let R be a G −graded ring and M a graded R −module. Then, M is a graded multiplication module if and only if for each $L \in GMax(R)$ with $M_L \neq 0_L$, M is $(L \cap h(R))$ −cyclic.*

Proof. M is graded multiplication module if and only if M is graded $(L \cap h(R))$ −multiplication module for each $L \in GMax(R)$ with $M_L \neq 0_L$. But for $M_L \neq 0_L$, M is graded $(L \cap h(R))$ −multiplication module if and only if M is graded $(L \cap h(R))$ −cyclic.

Theorem 3.17 *Let R be a G − graded ring, L a graded maximal ideal of R. A graded finitely generated R −module M is graded $(L \cap h(R))$ −cyclic if and only if M_L is graded cyclic.*

Proof. The first inclusion is clear since it holds for any graded R −module. For the converse, assume that M is graded finitely generated R −module such that M_L is graded cyclic. Then, $M_L = R_Lm$ for some $m \in h(M)$. Since M is graded finitely generated, then there exists $x \in h(R) - L$ with $xM \subseteq Rm$. Thus, M is graded $(L \cap h(R))$ −cyclic.

Theorem 3.18 *Let R be a G − graded ring and M a graded R −module. Then, M is a graded $(L \cap h(R))$ −cyclic module for each $L \in GMax(R)$ if and only if M is a graded finitely generated multiplication module.*

Proof. Assume that M is a graded $(L \cap h(R))$ −cyclic module for each graded maximal ideal L of R. Then, M is a graded multiplication module. Now, for each graded maximal ideal L_i, choose $x_i \in h(R) - L_i$ and $m_i \in h(M)$ such that $x_iM \subseteq Rm_i$. Note that $(\{x_i : x_i \in h(R) - L_i\}) = R$ and thus there exist $i_1, i_2, \dots, i_n$ with $(x_{i_1}, \dots, x_{i_n}) = R$. Thus, $M = RM = (x_{i_1}, \dots, x_{i_n})M \subseteq x_{i_1}M + \dots + x_{i_n}M \subseteq Rm_{i_1}$. Thus, $M = Rm_{i_1} + \dots + Rm_{i_n}$ is graded finitely generated module. Conversely; assume that M is a graded finitely generated multiplication module. For each graded maximal ideal L of R, we have M_L is graded cyclic and thus M is graded L − cyclic.

Now, we recall the following definition.

Definition 3.19 *[7] Let R be a G − graded ring, M a graded R −module and $N, N_1, N_2, \dots, N_n$ graded submodules of M. Then, the covering $N \subseteq \bigcup_{j=1}^{n} N_j$ is called efficient if for each $k \in \{1, 2, \dots, n\}$, we have $N \nsubseteq \bigcup_{j \neq k} N_j$.*

The following result is the graded S −version of the Prime Avoidance Lemma for graded S −multiplication modules.

Theorem 3.20 *Let R be a G −graded ring, $S \subseteq h(R)$ a multiplicatively closed subset of R, M a graded S −multiplication module and $N, N_1, N_2, \dots, N_n$ graded submodules of M such that $N \subseteq \bigcup_{j=1}^{n} N_j$ and at least $(n-2)$ of the graded submodules $N_1, \dots, N_n$ are graded S −prime. Then, there exists $x \in S$ with $xN \subseteq N_j$ for some $j \in \{1, \dots, n\}$.*

Proof. If $n = 1$, we are done. If $n = 2$, also we are done since if $N \subseteq N_1 \cup N_2$, then $N \subseteq N_1$ or $N \subseteq N_2$. Now, assume that $n \geq 3$ and suppose the result is true for all graded submodules N and covers of N by at most n graded submodules with at least $(n-2)$ of them being graded S −prime. Assume that $N \subseteq \bigcup_{j=1}^{n+1} N_j$. We may assume that $N \nsubseteq \bigcup_{j \neq k} N_j$ for each $k \in \{1,2,\dots,n+1\}$. **Case (1):** Suppose that there exist distinct $k, r \in \{1,2,\dots,n+1\}$ such that $x_1(N_k :_R M) \subseteq (N_r :_R M)$ for some $x_1 \in S$. M is graded S −multiplication modules implies that there exists $x_2 \in S$ with $x_2 N_k \subseteq (N_k :_R M)M$. Thus, for $x = x_1 x_2 \in S$, we have $xN_k = x_1(x_2 N_k) \subseteq x_1(N_k :_R M)M \subseteq (N_r :_R M)M \subseteq N_r$. Thus, $xN \subseteq \bigcup_{j=1}^{n+1} xN_j \subseteq \bigcup_{j \neq k} N_j$. Thus, by induction, there exists $x_3 \in S$ with $x_3(xN) \subseteq N_j$ for some j. Thus, for $\overline{x} = x_1 x_2 x_3 \in S$, we have $\overline{x}N \subseteq N_j$. **Case (2):** Suppose that for each distinct $k, r \in \{1,\dots,n+1\}$ and for each $x \in S$, we have $x(N_k :_R M) \nsubseteq (N_r :_R M)$. Suppose there does not exist $x \in S$ with $xN \subseteq N_j$ for some j. Thus, since M is a graded S −multiplication module, we have $x(N :_R M) \nsubseteq (N_j :_R M)$ for any $x \in S$. Since $n \geq 3$, then some N_j is graded S −prime which without loss of generality, we may take to be N_{n+1}. Note that $(N_{n+1} :_R M) \cap S = \phi$ by ([16], Proposition 2.9), we have $(N_{n+1} :_R M)$ is graded S −prime ideal. Claim that $N \cap (\bigcap_{j=1}^{n} N_j) \subseteq N_{n+1}$. To prove the claim, let $y \in N \cap (\bigcap_{j=1}^{n} N_j)$. Since the covering is efficient, then there exists $t \in (N \cap h(M)) - (\bigcup_{j=1}^{n} N_j)$, so $t \in N_{n+1}$. Now, $y + t \in N$, so $y + t \in \bigcup_{j=1}^{n+1} N_j$. If $y + t \in N_j$ for some j with $1 \leq j \leq n$, then $t = (y+t) - y \in N_j$, a contradiction. Thus, $y + t \in N_{n+1}$. But, then $y = (y+t) - t \in N_{n+1}$. Thus, $N \cap (\bigcap_{j=1}^{n} N_j) \subseteq N_{n+1}$. Thus, $(N :_R M) \prod_{j=1}^{n} (N_j :_R M) \subseteq (N :_R M) \cap (\bigcap_{j=1}^{n} (N_j :_R M))$. Since $(N_{n+1} :_R M)$ is a graded S −prime ideal, by ([16], Corollary 2.6), then there exists $x \in S$ with $x(N :_R M) \subseteq (N_{n+1} :_R M)$ or $x(N_j :_R M) \subseteq (N_{n+1} :_R M)$ for some $j \in \{1,2,\dots,n\}$, a contradiction.

Theorem 3.21 *Let R be a G −graded ring, $S \subseteq h(R)$ a multiplicatively closed subset of R satisfying the maximal multiple condition and M a graded R −module. Then, M is a graded S −multiplication module if and only if for each graded cyclic submodule Rm of M, where $m \in h(M)$, there exist $x \in S$ and a graded ideal P of R with $xRm \subseteq PM \subseteq Rm$.*

Proof. The "only if" part is clear. For the converse, assume that for each graded cyclic submodule Rm of M, there exist $x \in S$ and a graded ideal P of R with $xRm \subseteq PM \subseteq Rm$. Now, take a graded submodule L of M. Suppose $L = \sum_{m_j \in L \cap h(M)} Rm_j$. Thus, by assumption, there exists $x_j \in S$ such that $x_j Rm_j \subseteq (Rm_j :_R M)M$. Thus, $xRm_j \subseteq (Rm_j :_R M)M$ and thus $xL = x \sum_{m_j \in L \cap h(M)} Rm_j = \sum_{m_j \in L \cap h(M)} xRm_j$

$$\subseteq \sum_{m_j \in L \cap h(M)} (Rm_j :_R M)M = (\sum_{m_j \in L \cap h(M)}$$

and thus $xL \subseteq (L :_R M)M \subseteq L$. Therefore, M is a graded S −multiplication module.

Corollary 3.22 *Let R be a G −graded ring and M a graded R −module. Then, M is a* *graded multiplication module if and only if for each $m \in h(M)$, we have $Rm = PM$ for some graded ideal P of R.*

Proof. Follows directly from Theorem 3.21 by taking $S = \{1_R\}$.

Recall the following definition.

Definition 3.23 *[10] Let R be a G −graded ring, $S \subseteq h(R)$ a multiplicatively closed subset of R and M a graded R −module. A graded submodule N of M is said to be a graded S −finite submodule if there exists a finitely generated graded submodule L of M such that $xN \subseteq L \subseteq N$ for some $x \in S$. Also, M is said to be a graded S −Noetherian module if each graded submodule is graded S −finite. In particular, R is said to be a graded S −Noetherian ring if it is a graded S −Noetherian R −module.*

Theorem 3.24 *Let R be a G −graded ring, $S \subseteq h(R)$ a multiplicatively closed subset of R and M a graded S −Noetherian R −module. Then, M is a graded S −multiplication module if and only if for each graded cyclic submodule Rm of M, where $m \in h(M)$, there exist $x \in S$ and a graded ideal P of R with $xRm \subseteq PM \subseteq Rm$.*

Proof. The "only if" part is clear. For the converse, assume that M is a graded S −Noetherian R −module and for each graded cyclic submodule Rm of M, there exist $x \in S$ and a graded ideal P of R with $xRm \subseteq PM \subseteq Rm$. Now, take a graded submodule L of M. Since M is a graded S −Noetherian R −module, then there exist $x \in S$ and a graded finitely generated submodule $N = \sum_{j=1}^{n} R\overline{m_i}$ such that $xL \subseteq N \subseteq L$, where $\overline{m_j} \in h(M)$. By hypothesis, there exists $\overline{x_j} \in S$ such that $\overline{x_j}R\overline{m_j} \subseteq (R\overline{m_j}:_R M)M$. Now, put $\overline{x} = \overline{x_1}\,\overline{x_2}\ldots\overline{x_n}$ and $\overline{\overline{x}} = x\overline{x}$. Then, $\overline{\overline{x}} \in S$ and $\overline{x}R\overline{m_j} \subseteq (R\overline{m_j}:_R M)M$. Thus, $\overline{\overline{x}}L \subseteq \overline{x}N \subseteq \sum_{j=1}^{n} \overline{x}R\overline{m_j} \subseteq \sum_{j=1}^{n} (R\overline{m_j}:_R M)M \subseteq (\sum_{j=1}^{n} R$ and thus $\overline{\overline{x}}L \subseteq (L:_R M)M$. Therefore, M is a graded S −multiplication module.

Theorem 3.25 *Let R be a G −graded ring, $S \subseteq h(R)$ a multiplicatively closed subset of R, M a graded S −multiplication R −module and L a graded S −finite submodule of M. Then, there exist $\overline{x} \in S$ and a graded finitely generated ideal P of R with $\overline{x}L \subseteq PM \subseteq L$.*

Proof. Assume that M is a graded S −multiplication R −module and L is a graded S −finite submodule of M. Then, there exist $x_1, x_2 \in S$ and a graded finitely generated submodule $N = \sum_{j=1}^{n} R\overline{m_j}$ such that $x_1L \subseteq (L:_R M)M \subseteq L$ and $x_2L \subseteq \sum_{j=1}^{n} R\overline{m_j} \subseteq L$. Put $x = x_1x_2 \in S$, then $xL \subseteq (L:_R M)M$ and also $xL \subseteq \sum_{j=1}^{n} R\overline{m_j} \subseteq L$. Thus, for each $1 \leq j \leq n$, we have $x\overline{m_j} \in (L:_R M)M$ and thus $x\overline{m_j} = r_{j_1}m_{j_1} + \ldots + r_{j_k}m_{j_k}$ for some $m_{j_k} \in h(M)$ and $r_{j_k} \in (L:_R M) \cap h(R)$. Put $\overline{x} = x^2$ and consider the graded finitely generated ideal $P = \sum_{j=1}^{n}\sum_{i=1}^{k} Rr_{j_i}$ of R. Thus, $PM \subseteq (L:_R M)M \subseteq L$. Let $m \in L \cap h(M)$. Then, $xm \in xL \subseteq \sum_{j=1}^{n} R\overline{m_j}$ and so $xm = x_1\overline{m_1} + \ldots + x_n\overline{m_n}$ for some $x_j \in h(R)$ ad thus, $\overline{x}m = x^2m = x(xm) = x(x_1\overline{m_1} + \ldots + x_n\overline{m_n}) = x_1x\overline{m_1} +$ $x_1(r_{1_1}m_{1_1} + \ldots + r_{1_k}m_{1_k}) + \ldots + x_n(r_{n_1}m_{n_1} + \ldots + r_{n_k}m_{m_k})$ and thus, $\overline{x}m \in PM$. Therefore, $\overline{x}L \subseteq PM \subseteq L$.

Corollary 3.26 *Let R be a G −graded ring, M a graded multiplication module and L a graded finitely generated submodule of M. Then, $L = PM$ for some graded finitely generated ideal P of R.*

Proof. Let $S = \{1_R\}$. Then, M is a graded S −multiplication module and L is a graded S −finite submodule. Now, the result follows from Theorem 3.25.

Theorem 3.27 *Let R be a G −graded S −Noetherian ring, where $S \subseteq h(R)$ a multiplicatively closed subset of R and M a graded S −finite R −module. If M is a graded S −multiplication module, then M is a graded S −Noetherian module.*

Proof. Assume that R is a graded S −Noetherian ring and M is a graded S −multiplication module which is graded S −finite submodule. Let L be graded submodule of M. Since R is a G −graded S −Noetherian ring, then there exist $x_1,\dots,x_n \in (L:_R M) \cap h(R)$ such that $x(L:_R M) \cap h(R) \subseteq \sum_{j=1}^{n} Rx_j$. As M is graded S −multiplication module, then there exists $\bar{x} \in S$ with $\bar{x}L \subseteq (L:_R M)M$. Since M is graded S −finite, then there exist $m_1,\dots,m_r \in h(M)$ and $\bar{\bar{x}} \in S$ with $\bar{\bar{x}}M \subseteq \sum_{j=1}^{r} Rm_j$. Now, put $x^* = x\,\bar{x}\,\bar{\bar{x}} \in S$. Then, we have $x^* L \subseteq (x(L:_R M) \cap h(R))(\bar{\bar{x}}M) \subseteq (\sum_{j=1}^{n} Rx_j)(\sum_{i=1}^{r} Rm_i) = \sum_{i=1}^{n}\sum_{i=1}^{r} Rx_j m_i$. Also note that $x_j m_i \in L$. Thus, L is a graded S −finite and hence M is a graded S −Noetherian module.

Let R_1 and R_2 be G −graded rings. As in [8], $R = R_1 \times R_2$ is a G −graded ring with $R_g = (R_1)_g \times (R_2)_g$ for all $g \in G$. Let M_1 be a G −graded R_1 −module, M_2 be a G −graded R_2 −module and $R = R_1 \times R_2$. Then $M = M_1 \times M_2$ is a G −graded R −module with $M_g = (M_1)_g \times (M_2)_g$ for all $g \in G$. Also, if $S_1 \subseteq h(R_1)$ is a multiplicatively closed subset of R_1 and $S_2 \subseteq h(R_2)$ is a multiplicatively closed subset of R_2, then $S = S_1 \times S_2$ is a multiplicatively closed subset of R. Furthermore, each graded submodule of M is of the form $N = N_1 \times N_2$, where N_j is a graded submodule of M_j for $j = 1, 2$.

Theorem 3.28 *Let R_j be a G −graded ring, M_j a graded R_j module and $S_j \subseteq h(R_j)$ a multiplicatively closed subset of R_j for each $j \in \{1, 2\}$. Suppose that $M = M_1 \times M_2$ be a graded $R = R_1 \times R_2$ −module and $S = S_1 \times S_2$ be a multiplicatively closed subset of R. If M is a graded S −multiplication R −module, then M_1 is a graded S_1 −multiplication R_1 −modue and M_2 is a graded S_2 −multiplication R_2 −modue.*

Proof. Assume that M is a graded S −multiplication R −module. Without loss of generality we will show that M_1 is a graded S_1 −multiplication R_1 −modue. Let K_1 be a graded submodule of M_1. Then, $K_1 \times \{0\}$ is a graded submodule of M. Since M is a graded S −multiplication R −module, then there exist $x = (x_1, x_2) \in S_1 \times S_2$ such that $(x_1, x_2)(K_1 \times \{0\}) \subseteq ((K_1 \times \{0\}:_R M))M$. Thus, $x_1 K_1 \subseteq (K_1:_{R_1} M_1)M_1$. Thus, M_1 is a graded S_1 −multiplication R_1 −modue.

Theorem 3.29 *Let R_j be a G −graded ring, M_j a graded R_j −module and $S_j \subseteq h(R_j)$ a multiplicatively closed subset of R_j for each $j \in \{1,2,\dots,n\}$. Suppose that $M = M_1 \times M_2 \times\dots\times M_n$ be a graded $R = R_1 \times R_2 \times\dots\times R_n$ −module and $S = S_1 \times S_2 \times\dots\times S_n$ a multiplicatively closed subset of R. If M is a graded S −multiplication R −module, then M_j is a graded S_j −multiplication R_j −modue for each $j \in \{1,2,\dots,n\}$.*

Proof. Use induction on n.

4 Conclusion

Here, we represented a new form of the theory of graded rings. We constructed more accurate results and concepts regarding generalizations of graded multiplication modules. We investigated the relations between graded S −multiplication modules and graded $S-$cyclic modules. Furthermore, we proved if M is a graded $S-$multiplication module, then M is a graded $S-$Noetherian module. We can generalize the notion of graded $S-$multiplication modules to the notion of graded $S-$multiplication $2-$absorbing modules in the next work.

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