Al-Aqsa University Journal (Natural Sciences Series),Vol.23, Pages 31 -49, Jan.2022
ISSN 2070-3155

Performance of Some New Biasing Parameter Estimates of Kibria-Lukman Estimator in Linear Regression Model

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أداء بعض التقديرات الجديدة لمعلمة التحيز لمقدر Kibria–Lukman في نموذج الانحدار الخطي

الملخص

هدفت هذه الدراسة إلى اقتراح واختبار بعض المقدرات الجديدة لمعلمة التحيز في المقدر الذي تم اقتراحه مؤخراً وسمي بمقدر Kibria–Lukman (KL) لحل مشكلة التداخل الخطي المتعدد بين المتغيرات المستقلة في نموذج الانحدار الخطي بناءً على تقنيات التقدير المقترحة في الدراسات السابقة لمعلمة التحيز في مقدر الريدج نظراً لأهمية الاختيار المناسب لمعلمة التحيز في هذا المقدر. حيث أنه تم إجراء دراسة محاكاة وتطبيق على بيانات واقعية لتقييم أداء المقدرات المقترحة لمعلمة التحيز في مقدر KL بناء على معيار متوسط مربع الخطأ. وبالتالي تم تحديد أفضل المقدرات أداءً لمعلمة التحيز.

Abstract

The objective of this study is to propose and examine some new estimators of estimating the biasing parameter in the estimator that was recently proposed and called the Kibria-Lukman (KL) estimator for combating the multicollinearity problem between the explanatory variables in the linear regression model based on the proposed estimation techniques of the biasing parameter in the ridge estimator because of the importance of the useful selection of the biasing parameter in this estimator. The simulation study and the real-life data have been performed to evaluate the performance of the proposed methods due to the mean squared error. Then, we determine the best-performed biasing parameter estimators.

Keywords: Linear regression; Least Squares; multicollinearity; Kibria-Lukman estimator; biasing parameter; MSE

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1. Introduction:

The matrix form of the linear regression model is given as

$$y = X\beta + \varepsilon\,, \tag{1}$$

where y is an $n\times 1$ responses vector, X is a known $n\times p$ explanatory variables full rank matrix, β is an $p\times 1$ unknown regression coefficients vector, and ε is an $n\times 1$ errors vector with mean equals zero and variance-covariance matrix, $Cov(\varepsilon) = \sigma^2 I_n$, I_n is an $n\times n$ identity matrix. Then, the ordinary least squares estimator (OLS) of the parameter β in (1) is given as

$$\hat{\beta} = S^{-1}X'y\,, \tag{2}$$

where $S = X'X$.

In real-life situations, the multicollinearity problem often occurs for the linear regression model and is defined as the independence reduction or the interdependence existence among the columns of X (explanatory variables) with significant correlations (Farrar and Glauber, 1967), (Gunst and Måson, 1977), (Mason et al., 1975), and (Gunst, 1983). If the existence of the multicollinearity is perfect or severe, the OLS estimator's uniqueness does not hold (Belsley et al. 1980). Also, the OLS estimators are not efficient and unstable when severe multicollinearity exists. Moreover, this issue often gives us incorrect signs of the estimates of regression coefficients with non-significant and unexpected magnitude values, large standard errors with low *t*-test values, *R*-squared value near to one, and a large condition number (Ullah et al. 2019). Because of that many different estimators of the regression parameter have been gotten in order to cope with the multicollinearity problem. The most common regression estimator for solving this problem is called the ordinary ridge regression (ORR) estimator (Hoerl and Kennard, 1970). Then, Kibria and Lukman (2020) have recently proposed a new one-parameter ridge-type estimator and called the Kibria- Lukman (KL) estimator (Kibria and Lukman, 2020).

The new KL estimator of β (Kibria and Lukman, 2020) is given as

$$\hat{\beta}_{KL} = (S + k\mathrm{I}_p)^{-1}(S - k\mathrm{I}_p)\hat{\beta}, \qquad k \geq 0 \tag{3}$$

where k is called as the biasing parameter.

The biasing parameter selection is an issue with an important role in knowing the performance of the KL estimator similar to any biased regression estimator. Because of that, many studies are done to handle the problem of choosing a useful biasing parameter in the linear regression model, to mention a few of these studies: (Hoerl and Kennard, 1970), (Hoerl et al., 1975), (Lawless and Wang, 1976), (Hocking et al., 1976), (Kibria, 2003), (Khalaf and Shukur, 2005), (Alkhamisi et al., 2006), (Alkhamisi and Shukur, 2008), (Muniz and Kibria, 2009) and (Muniz et al., 2012), among others. Therefore, we propose and examine different methods for estimating the biasing parameter k of the KL estimator in the linear regression model following the techniques were proposed of k in the ORR estimator. The sections of this paper are arranged as follows. In Sec. 2, we give the statistical methodology. In Sec. 3, we give the different proposed estimators for the biasing parameter. A Monte Carlo simulation study and real-life data are done in Sec. 4. At last, some concluding remarks are presented in Sec. 5.

2. Statistical methodology

The canonical form of (1) (Scheffe, 1959) is given by

$$y = Z\alpha + \varepsilon, \tag{4}$$

where $Z = X\,G$ and $\alpha = G'\beta$. Here, G is an orthogonal matrix, G' is the transpose of G and $G'G = GG' = I$ such that $Z'Z = G'X'X\,G = C = diag(c_1, c_2, \ldots, c_p)$. The OLS estimator of α (Scheffe, 1959) is given as

$$\hat{\alpha} = C^{-1}Z'y. \tag{5}$$

The matrix form of the mean squared error (MMSE) and the mean squared error (MSE) of the OLS are given by

$$MMSE(\hat{\alpha}) = Cov(\hat{\alpha}) = \sigma^2 C^{-1} \tag{6}$$

and

$$MSE(\hat{\alpha}) = trace(MMSE(\hat{\alpha})) = \sigma^2 \sum_{i=1}^{p} \frac{1}{c_i} . \tag{7}$$

The KL estimator of α (Kibria and Lukman, 2020) is given as

$$\hat{\alpha}_{KL} = WM\ \hat{\alpha}, \tag{8}$$

where $W = [I_p + k\,C^{-1}]^{-1},\ M = [I_p - k\,C^{-1}],\ k = \dfrac{\sigma^2}{2\alpha_i^2 + (\sigma^2 / c_i)}$,

and the MMSE and the MSE are given as

$$\begin{aligned} MMSE(\hat{\alpha}_{KL}) &= Cov(\hat{\alpha}_{KL}) + Bias(\hat{\alpha}_{KL})\,Bias(\hat{\alpha}_{KL})' \\ &= \sigma^2\, WM\, C^{-1} M' W' + [WM - I_p]\, \alpha\alpha' [WM - I_p]' \end{aligned} \tag{9}$$

and

$$\begin{aligned} MSE(\hat{\alpha}_{KL}) &= trace(MMSE(\hat{\alpha}_{KL})) \\ &= \sigma^2 \sum_{i=1}^{p} \frac{(c_i - k)^2}{c_i\,(c_i + k)^2} + 4k^2 \sum_{i=1}^{p} \frac{\alpha_i^2}{(c_i + k)^2}. \end{aligned} \tag{10}$$

The explanation of the existence of Equation (10) is as follows:

$$\begin{aligned} MSE(\hat{\alpha}_{KL}) &= trace(MMSE(\hat{\alpha}_{KL})) \\ &= trace(\sigma^2\, WM\, C^{-1} M' W' + [WM - I_p]\, \alpha\alpha' [WM - I_p]') \\ &= trace(\sigma^2 A) + trace(B) \\ &= \sigma^2 \sum_{i=1}^{p} \frac{(c_i - k)^2}{c_i\,(c_i + k)^2} + 4k^2 \sum_{i=1}^{p} \frac{\alpha_i^2}{(c_i + k)^2}. \end{aligned}$$

where

$$W = (I + k\,C^{-1})^{-1} = diag\left(\frac{c_1}{c_1 + k}, \frac{c_2}{c_2 + k}, \cdots, \frac{c_p}{c_p + k} \right)$$

$$M = (I - kC^{-1}) = diag\left(\frac{c_1 - k}{c_1}, \frac{c_2 - k}{c_2}, \cdots, \frac{c_p - k}{c_p}\right)$$

$$C^{-1} = diag\left(\frac{1}{c_1}, \frac{1}{c_2}, \cdots, \frac{1}{c_p}\right)$$

$$A = WM\,C^{-1}M'W'$$
$$= diag\left(\frac{(c_1 - k)^2}{c_1\,(c_1 + k)^2}, \frac{(c_2 - k)^2}{c_2\,(c_2 + k)^2}, \cdots, \frac{(c_p - k)^2}{c_p\,(c_p + k)^2}\right)$$

and

$$B = [WM - I_p]\,\alpha\,\alpha'[WM - I_p]'$$
$$= 4k^2\begin{pmatrix} \frac{\alpha_1^2}{(c_1 + k)^2} & \frac{\alpha_1\alpha_2}{(c_1 + k)(c_2 + k)} & \cdots & \frac{\alpha_1\alpha_p}{(c_1 + k)(c_p + k)} \\ \frac{\alpha_2\alpha_1}{(c_2 + k)(c_1 + k)} & \frac{\alpha_2^2}{(c_2 + k)^2} & \cdots & \frac{\alpha_2\alpha_p}{(c_2 + k)(c_p + k)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\alpha_p\alpha_1}{(c_p + k)(c_1 + k)} & \frac{\alpha_p\alpha_2}{(c_p + k)(c_2 + k)} & \cdots & \frac{\alpha_p^2}{(c_p + k)^2} \end{pmatrix}.$$

3. Proposed Estimators of the biasing parameter.

Different methods of estimating the biasing parameter k in the common ORR estimator are given in lots of previous studies. So, here following the estimation techniques were proposed of k in the ORR estimator by many authors, we propose different estimators of the biasing parameter $k = \dfrac{\sigma^2}{2\alpha_i^2 + (\sigma^2 / c_i)}$ of the KL estimator.

1. Following (Hoerl and Kennard, 1970) method, $\hat{k}_1$ of the KL estimator is given as

$$\hat{k}_1 = \min\left\{\frac{\hat{\sigma}^2}{2\hat{\alpha}_i^2 + (\hat{\sigma}^2 / c_i)}\right\}.$$

2. Following (Hoerl et al., 1975) method, $\hat{k}_2$ of the KL estimator is given as

$$\hat{k}_2 = \frac{p\hat{\sigma}^2}{\sum_{i=1}^{p}\left(2\hat{\alpha}_i^2 + (\hat{\sigma}^2 / c_i)\right)}.$$

3. Following (Lawless and Wang, 1976) method, $\hat{k}_3$ of the KL estimator is given as

$$\hat{k}_3 = \frac{p\hat{\sigma}^2}{\sum_{i=1}^{p}\left(2c_i\hat{\alpha}_i^2 + \hat{\sigma}^2\right)}.$$

4. Following (Hocking et al., 1976) method, $\hat{k}_4$ of the KL estimator is given as

$$\hat{k}_4 = \frac{\hat{\sigma}^2\sum_{i=1}^{p}\left(2c_i\hat{\alpha}_i + \hat{\sigma}^2\right)^2}{\left(\sum_{i=1}^{p}\left(2c_i\hat{\alpha}_i^2 + \hat{\sigma}^2\right)\right)^2}.$$

5. Following (Kibria, 2003) methods, $\hat{k}_5$, $\hat{k}_6$, and $\hat{k}_7$ of the KL estimator are given as

$$\hat{k}_5 = \frac{1}{p}\sum_{i=1}^{p}\frac{\hat{\sigma}^2}{\left(2\hat{\alpha}_i^2 + (\hat{\sigma}^2 / c_i)\right)}$$

$$\hat{k}_6 = \frac{\hat{\sigma}^2}{\left(\prod_{i=1}^{p}\left(2\hat{\alpha}_i^2 + (\hat{\sigma}^2 / c_i)\right)\right)^{\frac{1}{p}}}$$

$$\hat{k}_7 = Median\left\{ \frac{\hat{\sigma}^2}{\left(2\hat{\alpha}_i^2 + (\hat{\sigma}^2 / c_i)\right)} \right\}.$$

6. Following (Khalaf and Shukur, 2005) method, $\hat{k}_8$ of the KL estimator is given as

$$\hat{k}_8 = \frac{c_{\max}\hat{\sigma}^2}{\left((n-p+1)\hat{\sigma}^2 + 2c_{\max}\hat{\alpha}_{\max}^2\right)}.$$

7. Following (Alkhamisi et al., 2006) methods, $\hat{k}_9$, $\hat{k}_{10}$, and $\hat{k}_{11}$ of the KL estimator are given as

$$\hat{k}_9 = \frac{1}{p}\sum_{i=1}^{p} L_i$$

where $L_i = \dfrac{c_i\,\hat{\sigma}^2}{\left((n-p+1)\hat{\sigma}^2 + 2c_i\,\hat{\alpha}_i^2\right)}$.

$$\hat{k}_{10} = \max(L_i).$$

$$\hat{k}_{11} = Median\,(L_i).$$

8. Following (Muniz and Kibria, 2009) methods, $\hat{k}_{12}$, $\hat{k}_{13}$, $\hat{k}_{14}$, $\hat{k}_{15}$, $\hat{k}_{16}$, $\hat{k}_{17}$ and $\hat{k}_{18}$ of the KL estimator are given as

$$\hat{k}_{12} = \left(\prod_{i=1}^{p} L_i\right)^{\frac{1}{p}}.$$

$$\hat{k}_{13} = \max\left(\frac{1}{m_i}\right),$$

where $m_i = \sqrt{\dfrac{\hat{\sigma}^2}{\left(2\hat{\alpha}_i^2 + (\hat{\sigma}^2 / c_i)\right)}}$.

$$\hat{k}_{14} = \max(m_i) .$$

$$\hat{k}_{15} = \left(\prod_{i=1}^{p} \frac{1}{m_i} \right)^{\frac{1}{p}} .$$

$$\hat{k}_{16} = \left(\prod_{i=1}^{p} m_i \right)^{\frac{1}{p}} .$$

$$\hat{k}_{17} = Median\,(\frac{1}{m_i}) .$$

$$\hat{k}_{18} = Median\,(m_i) .$$

9. Following (Muniz et al., 2012) methods, $\hat{k}_{19}$, $\hat{k}_{20}$, $\hat{k}_{21}$, $\hat{k}_{22}$ andKL the of $\hat{k}_{23}$ estimator are given as

$$\hat{k}_{19} = Max\,(\frac{1}{F_i}) ,$$

where $F_i = \sqrt{\dfrac{c_{\max}\hat{\sigma}^2}{\left((n-p+1)\hat{\sigma}^2 + 2c_{\max}\hat{\alpha}_i^2\right)}}$.

$$\hat{k}_{20} = \left(\prod_{i=1}^{p} \frac{1}{F_i} \right)^{\frac{1}{p}} .$$

$$\hat{k}_{21} = Max\,(F_i) .$$

$$\hat{k}_{22} = \left(\prod_{i=1}^{p} F_i \right)^{\frac{1}{p}}.$$

$$\hat{k}_{23} = Median\,(\frac{1}{F_i})$$

4. Applications

4.1 Simulation Study

Because the theoretical comparisons are impossible to be performed in this case, a Monte Carlo simulation study has been done in order to show the biasing parameter new proposed estimators' performances of the KL estimator. We use the MSE criterion to measure the effect of the mentioned proposed estimators. The simulation study design depends on some factors that give a useful view of the proposed estimators' performances. We use MATLAB software for computational procedures. Following (Gibbons, 1981) and (Kibria, 2003), the explanatory variables are given by using equation (11):

$$x_{ij} = (1-\rho^2)^{1/2} z_{ij} + \rho z_{i,\,p+1}, \qquad i = 1,2,\ldots,n, \quad j = 1,2,\ldots,p \tag{11}$$

where z_{ij} are independent numbers follows a standard normal distribution and here $\rho = 0.90$ and $\rho = 0.99$ which is defined as the correlation of any two explanatory variables. We consider $p = 3$ and $p = 7$ in this simulation. These variables are in a standardized form. The n observations of the response variable y_i are obtained by equation (12):

$$y_i = \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_p x_{ip} + e_i, \quad i = 1,2,\ldots,n \tag{12}$$

where e_i are $i.i.d\; N(0,\sigma^2)$. We select β values where $\beta'\beta = 1$ (Newhouse and Oman, 1971). This simulation replications number is 1000 times for the given sample sizes $n = 50$ and 100 and for $\sigma^2 = 1$, 25, and 100. In each replicate, the estimators estimated MSE values are measured as:

$$MSE(\alpha^*) = \frac{1}{1000}\sum_{i=1}^{1000}(\alpha_i^* - \alpha)'(\alpha_i^* - \alpha) \qquad (13)$$

where α^* would be any of the estimators.

According to the results stated in Tables 1 - 4, we see the following:

1. The increase of the factors σ, ρ and p gives an increase of the estimated MSE values while the increase of the sample size n gives a decrease of the estimated MSE values.
2. The OLS estimator has the highest estimated MSE values among all biasing parameter proposed estimators of the KL estimator in the occurrence of the multicollinearity problem.
3. As a general result, the KL estimator with the biasing parameter proposed estimators gives better results with lower estimated MSE values than the OLS estimator. So, our biasing parameter proposed estimators have a useful impact on the KL estimator performance.
4. As an overall evaluation, the proposed estimators, $\hat{k}_5$, $\hat{k}_6$, $\hat{k}_7$, $\hat{k}_{14}$, $\hat{k}_{21}$, $\hat{k}_{22}$, $\hat{k}_{13}$, $\hat{k}_{19}$, $\hat{k}_{16}$ and $\hat{k}_{18}$ for the KL estimator give smaller estimated MSE values than the remaining of the proposed estimators of k.
5. Selecting the biasing parameter estimator $\hat{k}$ depends mostly on the collinearity level among the explanatory variables.
6. The KL estimator with the not ordered biasing parameter proposed estimators $\hat{k}_5$, $\hat{k}_6$, $\hat{k}_7$ and $\hat{k}_{14}$ is better than with others for $\rho = 0.90$ and different values of σ, n, and p.
7. The KL estimator with the not ordered biasing parameter proposed estimators $\hat{k}_{21}$, $\hat{k}_{22}$, $\hat{k}_{13}$, $\hat{k}_{19}$, $\hat{k}_{16}$ and $\hat{k}_{18}$ is better than with others for $\rho = 0.99$ and different values of σ, n, and p.

Table 1. Estimated MSE values for KL when $n = 50$ and $p = 3$

ρ	σ	OLS	$KL(\hat{k}_1)$	$KL(\hat{k}_2)$	$KL(\hat{k}_3)$	$KL(\hat{k}_4)$	$KL(\hat{k}_5)$
0.90	1	0.2136	0.1719	0.1422	0.2096	0.1192	0.0824
	5	5.3394	2.6439	1.6518	5.0401	3.9009	1.0271
	10	21.357	9.7561	6.1424	20.098	14.797	3.7387
0.99	1	1.9452	1.0538	0.6211	1.6563	1.4113	0.4501
	5	48.628	20.136	12.275	41.507	34.572	9.4351
	10	194.51	79.510	48.701	165.98	133.07	37.686
$\hat{k}$ continued							
ρ	σ	$KL(\hat{k}_6)$	$KL(\hat{k}_7)$	$KL(\hat{k}_8)$	$KL(\hat{k}_9)$	$KL(\hat{k}_{10})$	$KL(\hat{k}_{11})$
0.90	1	0.1062	0.1181	0.1783	0.1753	0.1396	0.1980
	5	1.1436	1.1721	3.4955	4.1452	2.9670	4.9261
	10	4.3990	5.0846	13.575	16.545	11.780	19.703
0.99	1	0.3881	0.3167	1.0243	0.9359	0.5664	1.7899
	5	8.7551	11.224	18.942	22.669	12.223	44.744
	10	35.105	45.752	74.712	90.503	48.564	178.97
$\hat{k}$ continued							
ρ	σ	$KL(\hat{k}_{12})$	$KL(\hat{k}_{13})$	$KL(\hat{k}_{14})$	$KL(\hat{k}_{15})$	$KL(\hat{k}_{16})$	$KL(\hat{k}_{17})$
0.90	1	0.1890	0.1160	0.0763	0.1531	0.1238	0.1473
	5	4.5974	3.7681	1.4712	4.2830	2.2786	4.2689
	10	18.370	15.363	5.7507	17.166	9.0316	16.932
0.99	1	1.5451	0.4664	0.1938	0.1569	0.1161	0.1338
	5	38.390	8.5639	3.8245	3.6124	2.5840	4.6049
	10	153.51	33.738	15.418	14.475	10.357	18.775
$\hat{k}$ continued							
ρ	σ	$KL(\hat{k}_{18})$	$KL(\hat{k}_{19})$	$KL(\hat{k}_{20})$	$KL(\hat{k}_{21})$	$KL(\hat{k}_{22})$	$KL(\hat{k}_{23})$
0.90	1	0.1296	0.1100	0.1352	0.1220	0.1443	0.1356
	5	2.2968	3.4401	3.7350	2.9433	3.2098	3.7871
	10	9.4686	13.967	14.981	11.726	12.783	15.057
0.99	1	0.0963	0.4523	0.1293	0.0766	0.0929	0.1138
	5	3.4846	8.0837	2.9907	1.5117	2.1397	4.1079
	10	14.230	31.817	11.977	5.9555	8.5682	16.764

Table 2: Estimated MSE values for KL when $n = 100$ and $p = 3$

ρ	σ	OLS	$KL(\hat{k}_1)$	$KL(\hat{k}_2)$	$KL(\hat{k}_3)$	$KL(\hat{k}_4)$	$KL(\hat{k}_5)$
0.90	1	0.1064	0.0926	0.0835	0.1061	0.0819	0.0612
	5	2.6611	1.4261	0.8945	2.5916	1.3759	0.6261
	10	10.644	4.9692	3.1477	10.326	4.8587	2.2893
0.99	1	0.9913	0.6133	0.3881	0.9278	0.6874	0.2519
	5	24.782	10.665	6.5734	22.471	13.944	5.9218
	10	99.128	41.866	25.890	90.114	53.202	22.403
$\hat{k}$ continued							
ρ	σ	$KL(\hat{k}_6)$	$KL(\hat{k}_7)$	$KL(\hat{k}_8)$	$KL(\hat{k}_9)$	$KL(\hat{k}_{10})$	$KL(\hat{k}_{11})$
0.90	1	0.0750	0.0784	0.0947	0.0960	0.0844	0.1022
	5	0.5930	0.6066	1.9166	2.1949	1.6321	2.5408
	10	2.2169	2.7148	7.3884	8.7423	6.4449	10.161
0.99	1	0.2234	0.2146	0.6077	0.4955	0.2668	0.9456
	5	4.7383	6.1399	10.258	11.119	5.3649	23.622
	10	18.756	25.105	40.158	44.343	21.155	94.485
$\hat{k}$ continued							
ρ	σ	$KL(\hat{k}_{12})$	$KL(\hat{k}_{13})$	$KL(\hat{k}_{14})$	$KL(\hat{k}_{15})$	$KL(\hat{k}_{16})$	$KL(\hat{k}_{17})$
0.90	1	0.0998	0.0820	0.0656	0.0906	0.0813	0.0890
	5	2.4293	2.3408	0.9930	2.4659	1.4545	2.4602
	10	9.7059	9.4753	3.8618	9.8857	5.7132	9.7964
0.99	1	0.8563	0.0265	0.1207	0.1004	0.1559	0.0981
	5	21.136	1.1227	3.1808	3.1206	3.3303	2.1878
	10	84.526	4.6562	11.896	12.360	13.298	8.4630
$\hat{k}$ continued							
ρ	σ	$KL(\hat{k}_{18})$	$KL(\hat{k}_{19})$	$KL(\hat{k}_{20})$	$KL(\hat{k}_{21})$	$KL(\hat{k}_{22})$	$KL(\hat{k}_{23})$
0.90	1	0.0830	0.0802	0.0866	0.0792	0.0862	0.0861
	5	1.4681	2.2430	2.3065	1.8791	1.9574	2.3207
	10	6.0591	9.0520	9.2448	7.4901	7.7930	9.2563
0.99	1	0.1555	0.0259	0.0823	0.0584	0.1634	0.1169
	5	4.3047	1.2167	2.6240	1.1617	3.4618	2.5000
	10	17.511	5.1163	10.556	4.2955	13.754	9.4103

Table 3: Estimated MSE values for KL when $n = 50$ and $p = 7$

ρ	σ	OLS	$KL(\hat{k}_1)$	$KL(\hat{k}_2)$	$KL(\hat{k}_3)$	$KL(\hat{k}_4)$	$KL(\hat{k}_5)$
0.90	1	0.7436	0.5116	0.2986	0.7112	0.5093	0.1392
	5	18.588	9.4319	4.4020	17.114	7.5979	2.8549
	10	74.355	36.849	17.162	68.325	29.774	11.080
0.99	1	7.0388	3.7251	1.7378	5.7398	4.1736	1.2683
	5	175.97	86.592	39.874	142.91	98.609	30.102
	10	703.88	345.53	159.05	572.09	394.65	117.74
$\hat{k}$ continued							
ρ	σ	$KL(\hat{k}_6)$	$KL(\hat{k}_7)$	$KL(\hat{k}_8)$	$KL(\hat{k}_9)$	$KL(\hat{k}_{10})$	$KL(\hat{k}_{11})$
0.90	1	0.1805	0.1978	0.5104	0.4855	0.2177	0.6413
	5	2.6794	3.1161	9.2814	11.163	3.6094	15.896
	10	10.541	12.372	36.199	44.535	14.137	63.566
0.99	1	1.0310	1.1913	3.4257	3.0895	2.1927	6.0288
	5	24.133	28.538	77.705	74.173	52.271	150.568
	10	96.420	113.96	309.74	295.85	208.63	602.23
$\hat{k}$ continued							
ρ	σ	$KL(\hat{k}_{12})$	$KL(\hat{k}_{13})$	$KL(\hat{k}_{14})$	$KL(\hat{k}_{15})$	$KL(\hat{k}_{16})$	$KL(\hat{k}_{17})$
0.90	1	0.6199	0.3022	0.1318	0.4733	0.2763	0.4655
	5	15.242	9.4146	2.7767	12.736	5.6527	12.510
	10	60.939	38.103	10.920	51.037	22.467	50.081
0.99	1	5.6354	2.9319	1.4883	1.5221	0.5467	1.6204
	5	140.55	69.900	36.915	37.203	13.344	39.964
	10	562.20	279.17	146.42	148.84	53.314	159.74
$\hat{k}$ continued							
ρ	σ	$KL(\hat{k}_{18})$	$KL(\hat{k}_{19})$	$KL(\hat{k}_{20})$	$KL(\hat{k}_{21})$	$KL(\hat{k}_{22})$	$KL(\hat{k}_{23})$
0.90	1	0.2848	0.3029	0.4649	0.1526	0.2865	0.4716
	5	5.9406	9.4794	12.503	3.6436	5.9931	12.673
	10	23.680	38.380	50.119	14.530	23.821	50.719
0.99	1	0.5702	2.7822	1.0964	2.1815	0.6219	1.2349
	5	13.943	65.462	26.913	53.933	15.774	30.734
	10	55.702	261.28	107.56	216.45	63.131	123.40

Table 4: Estimated MSE values for KL when $n=100$ and $p=7$

ρ	σ	OLS	$KL(\hat{k}_1)$	$KL(\hat{k}_2)$	$KL(\hat{k}_3)$	$KL(\hat{k}_4)$	$KL(\hat{k}_5)$
0.90	1	0.3264	0.2651	0.1978	0.3233	0.1438	0.0976
	5	8.1601	4.7840	2.2621	7.9100	3.2610	1.0684
	10	32.640	18.245	8.4794	31.567	13.017	4.0040
0.99	1	3.0983	1.9423	0.9542	2.8489	1.9824	0.4938
	5	77.456	42.559	19.460	69.903	45.834	11.459
	10	309.82	169.04	77.266	279.48	184.89	45.466
$\hat{k}$ continued							
ρ	σ	$KL(\hat{k}_6)$	$KL(\hat{k}_7)$	$KL(\hat{k}_8)$	$KL(\hat{k}_9)$	$KL(\hat{k}_{10})$	$KL(\hat{k}_{11})$
0.90	1	0.1485	0.1635	0.2707	0.2881	0.1926	0.3101
	5	1.2887	1.3097	5.3959	6.8729	3.5174	7.7209
	10	4.8465	4.9668	20.995	27.423	13.823	30.879
0.99	1	0.5287	0.5547	1.8714	1.8816	0.8649	2.9314
	5	10.838	11.220	40.069	44.419	19.810	73.252
	10	43.039	44.735	158.88	177.06	78.884	293.00
$\hat{k}$ continued							
ρ	σ	$KL(\hat{k}_{12})$	$KL(\hat{k}_{13})$	$KL(\hat{k}_{14})$	$KL(\hat{k}_{15})$	$KL(\hat{k}_{16})$	$KL(\hat{k}_{17})$
0.90	1	0.3066	0.2522	0.1242	0.2870	0.2122	0.2840
	5	7.6041	6.9211	2.2940	7.5047	4.1819	7.5027
	10	30.406	27.865	8.9818	30.063	16.524	30.051
0.99	1	2.8525	0.1435	0.2904	0.2689	0.4084	0.2698
	5	71.147	3.3542	7.5694	7.7847	9.0032	7.7081
	10	284.55	13.486	30.265	31.352	35.822	30.892
$\hat{k}$ continued							
ρ	σ	$KL(\hat{k}_{18})$	$KL(\hat{k}_{19})$	$KL(\hat{k}_{20})$	$KL(\hat{k}_{21})$	$KL(\hat{k}_{22})$	$KL(\hat{k}_{23})$
0.90	1	0.2180	0.2491	0.2778	0.1911	0.2317	0.2779
	5	4.1709	6.7873	7.2167	4.6512	5.2000	7.2643
	10	16.498	27.303	28.903	18.592	20.701	29.094
0.99	1	0.4203	0.1421	0.4023	0.1314	0.2749	0.4306
	5	9.2310	3.6143	11.633	3.1711	5.7765	12.602
	10	36.860	14.611	46.817	12.729	22.925	50.748

4.2 Real-Life data

The Portland cement data (Woods et al., 1932) is used to examine the KL estimator performance with the different biasing parameter proposed estimators of k. Also, this data is used by many authors, to mention a few of them: (Kaciranlar et al., 1999); (Li and Yang, 2012); (Lukman et al., 2019) and recently (Kibria and Lukman, 2020). This data regression model is given as

$$y_i = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \beta_4 X_4 + \varepsilon_i . \tag{14}$$

To know more about this data and variables, see (Woods et al., 1932). Some measures are calculated to show the multicollinearity occurrence as the variance inflation factors (VIFs) which are as follows: $VIF_1 = 38.50$, $VIF_2 = 254.42$, $VIF_3 = 46.87$, $VIF_4 = 282.51$. Eigenvalues of S are $c_1 = 44676.206$, $c_2 = 5965.422$, $c_3 = 809.952$, $c_4 = 105.419$, and the condition number (CN) of S is approximately 20.58. The VIFs, the eigenvalues, and the CN all give us an indicator that severe multicollinearity occurs in this data (Dawoud and Kibria, 2020). In Table 5, we present the estimated coefficients and the estimators' MSE values. Table 5 showed that the KL estimator with the biasing parameter proposed estimators $\hat{k}_1$ and $\hat{k}_8$ outperforms better than with other proposed estimators of k. In particular, the KL estimator with $\hat{k}_1$ performs the best for this data.

Table 5: The results of MSE values for OLS and KL with different biasing estimators

Values	OLS	$KL(\hat{k}_1)$	$KL(\hat{k}_2)$	$KL(\hat{k}_3)$	$KL(\hat{k}_4)$
MSE	4912.0902	2170.960	7255.602	15438.307	20475.96
$\hat{k}$	-------	0.000471	0.002354	0.010152	115.6564
$\hat{k}$ continued					
Values	$KL(\hat{k}_5)$	$KL(\hat{k}_6)$	$KL(\hat{k}_7)$	$KL(\hat{k}_8)$	$KL(\hat{k}_9)$
MSE	20474.79	20452.469	20467.455	2579.539	20474.062
$\hat{k}$	86.20333	3.038513	11.332884	0.000768	70.435719
$\hat{k}$ continued					
Values	$KL(\hat{k}_{10})$	$KL(\hat{k}_{11})$	$KL(\hat{k}_{12})$	$KL(\hat{k}_{13})$	$KL(\hat{k}_{14})$

MSE	20478.395	20466.82	20442.811	20472.673	20469.29
$\hat{k}$	200.78711	10.19202	2.0599761	46.074340	16.57634
$\hat{k}$ continued					
Values	$KL(\hat{k}_{15})$	$KL(\hat{k}_{16})$	$KL(\hat{k}_{17})$	$KL(\hat{k}_{18})$	$KL(\hat{k}_{19})$
MSE	20365.396	20437.369	20266.06	20454.451	20471.931
$\hat{k}$	0.573679	1.743133	0.297050	3.366434	36.081046
$\hat{k}$ continued					
Values	$KL(\hat{k}_{20})$	$KL(\hat{k}_{21})$	$KL(\hat{k}_{22})$	$KL(\hat{k}_{23})$	
MSE	20360.16	20469.277	20439.020	20264.85	
$\hat{k}$	0.546925	16.499865	1.828401	0.295305	

5. Some Concluding Remarks

In the linear regression model, the KL estimator performance depends almost on the biasing parameter selection. Within different biasing parameter proposed estimators, $\hat{k}_1$ to $\hat{k}_{23}$, we focus on determining the ones that outperform better by a simulation study and a real-life data application. Then, we investigate in detail the different biasing parameter proposed estimators of the KL estimator by considering different levels of correlation, different values of error variance, different values of sample sizes, and different numbers of the explanatory variables in the simulation study. In general, the KL estimator is favorable with the different biasing parameter proposed estimators. The KL estimator with the biasing parameter proposed estimators, $\hat{k}_5$, $\hat{k}_6$, $\hat{k}_7$, $\hat{k}_{14}$, $\hat{k}_{21}$, $\hat{k}_{22}$, $\hat{k}_{13}$, $\hat{k}_{19}$, $\hat{k}_{16}$ and $\hat{k}_{18}$ is better than with the others in the simulation study. According to the real-life data, it is obvious that the KL estimator is better than the OLS estimator through the estimators $\hat{k}_1$ and $\hat{k}_8$ that give smaller MSE values. So, we can say that the KL estimator with at least one of the biasing parameter proposed estimators should provide better results than the OLS. Once for all, the KL estimator with the biasing parameter proposed estimators $\hat{k}_1$, $\hat{k}_5$, $\hat{k}_6$, $\hat{k}_7$, $\hat{k}_8$, $\hat{k}_{14}$, $\hat{k}_{21}$, $\hat{k}_{22}$, $\hat{k}_{13}$, $\hat{k}_{19}$ and $\hat{k}_{16}$ is recommended for the practitioners in the linear regression model.

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